Bounds on the mobility of electrons in weakly ionized plasmas

A. Rokhlenko
Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903

Joel L. Lebowitz
Department of Mathematics and Department of Physics, Rutgers University, New Brunswick, New Jersey 08903

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We obtain exact upper and lower bounds on the steady-state drift velocity, and kinetic energy of electrons, driven by an external field in a weakly ionized plasma (swarm approximation). The scattering is assumed to be elastic with a simplified velocity dependence of the collision cross sections. When the field is large the bounds are close to each other and to the results obtained from the conventional approximation of the Boltzmann equation in which one keeps only the first two terms of a Legendre expansion. The bounds prove rigorously that it is possible to increase the electron mobility by the addition of suitably chosen scatterers to the system as predicted by the Druyvesteyn approximation and found in experiments. [S1063-651X(97)07807-0]

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I. INTRODUCTION

The behavior of the electron mobility in a gas composed of several species is a subject of continued experimental and theoretical investigations [1–4]. Of particular interest is the fact that the *addition* of certain types of scatterers, i.e., neutral species, to the gas increases the electron mobility and therefore the electron current in an applied electric field [3,4]. This effect is potentially of practical utility and, as was pointed out by Nagpal and Garscadden [4], can be used to obtain information about scattering cross sections and level structure of different species.

The fact that the mobility can actually increase with the addition of scatterers is at first surprising: It is contrary to the well-known Matthiessen rule in metals, which states that the total resistivity due to different types of scatterers is the sum of resistivities due to each of them [5]. A closer inspection shows that Matthiessen's rule refers to the linear regime of small electric fields, while the observations and analysis in gases [3,4] are in the nonlinear high-field regime.

This still leaves open the question of the validity of approximations commonly made in calculating the current of weakly ionized plasmas in strong fields. We therefore investigate here rigorously the stationary solutions of the kinetic equation for the electron velocity distribution function in cases where the electron–neutral-atom (e-n) collisions are purely elastic and their cross section is modeled by a simple power dependence on the electron speed. In particular we establish two-sided bounds for the electron mean energy and drift in the presence of an external electric field. These bounds show that the results obtained for the current and energy of the electrons in the usual approximation, which neglects higher-order terms in a Legendre polynomial expansion and gives the Druyvesteyn-like distribution for large fields, are qualitatively right and even provide good quantitative answers. In fact, they are sufficiently precise to confirm an increase in the current for large (but not for small) fields upon addition of some gases, provided the mass of the added species is smaller than that of the dominant one, e.g., adding helium to a xenon gas, and the different cross sections satisfy certain conditions. We believe that our analysis can be extended to include more realistic elastic cross sections and inelastic collisions; these are most important in practice for enhancement of the electron mobility.

II. KINETIC EQUATION

Our starting point is the commonly used swarm approximation, applicable to gases with a very small degree of ionization [6–10]. In this approximation only e-n collisions are taken into account in the kinetic equation for the electron distribution function (EDF) $f(\mathbf{r},\mathbf{v},t)$. The neutral atoms themselves, which may consist of several species, are assumed to have a Maxwellian distribution with a specified common temperature T_n. Further simplification is achieved if the e-n collisions are assumed to be essentially elastic: the collision integral can then be reduced [1,6] to a differential operator due to the great difference in the masses of the electrons and neutrals. To simplify matters further we consider the case where the scattering is spherically symmetric. The stationary kinetic equation for the normalized EDF, in a spatially uniform system with constant density n subject to an external electric field F, can then be written in the form [6]

$$-\frac{e}{m}\mathbf{F}\cdot\boldsymbol{\nabla}_v f=\frac{1}{v^2}\frac{\partial}{\partial v}\left[\epsilon(v)\,\frac{v^4}{\lambda(v)}\left(f_0+\frac{kT_n}{mv}\frac{\partial f_0}{\partial v}\right)\right]+\frac{v}{\lambda(v)}\,(f_0-f\,),\tag{1}$$

$$\lambda(v)=\left[\sum_{i=1}^{S}N_i\sigma_i(v)\right]^{-1},\quad \epsilon(v)=\lambda(v)m\sum_{i=1}^{S}\frac{N_i\sigma_i(v)}{M_i}.$$

Here e,m are the electron charge and mass, σ_i is the collision cross section with species i whose mass is M_i and number density is N_i, λ is the mean free path in the e-n collisions, k is Boltzmann's constant, and f_0 is the spherically symmetric part of the distribution function,

$$f_0(v)=\frac{1}{4\pi}\int f(\mathbf{v})d\Omega.$$

We note that ϵ is a small parameter equal to the ratio of the electron mass to the mean mass of neutral scatterers $\epsilon=m\overline{M^{-1}}$, where $\overline{M^{-1}}=\Sigma M_i^{-1}N_i\sigma_i/\Sigma N_i\sigma_i$.

A. Velocity-independent cross sections

We shall consider first the case where $\sigma_i(v)$ is independent of v so $\lambda=\text{const}$ and $\epsilon=\text{const}$. Taking the electric field parallel to the z axis, Eq. (1) can be written in the dimensionless form

$$-E\frac{\partial f}{\partial u_z}=\epsilon\frac{1}{u^2}\frac{\partial}{\partial u}\left[u^4\left(f_0+\frac{T}{u}\frac{\partial f_0}{\partial u}\right)\right]+u(f_0-f\ ), \quad (2)$$

where

$$\mathbf{u}=\gamma\mathbf{v}, \quad u=\sqrt{u_x^2+u_y^2+u_z^2},$$

$$\gamma=\sqrt{\frac{m}{kT_0}}, \quad T=\frac{T_n}{T_0}, \quad E=\frac{e\lambda|\mathbf{F}|}{kT_0}$$

with some fixed T_0 specifying the units of the temperature. We normalize f so that

$$\frac{1}{4\pi}\int f(\mathbf{u})d^3u=\int_0^\infty u^2 f_0 du=1. \quad (3)$$

When $E=0$ the stationary distribution is the Maxwellian with temperature T,

$$f=f_0=M(u)=\sqrt{\frac{2}{\pi T^3}}\exp\left(\frac{-u^2}{2T}\right); \quad (4)$$

$M(u)$ is the unique solution of Eq. (2) for $E=0$, $\epsilon\neq 0$. When $E\neq 0$ the situation is more complicated. Only for E small compared to ϵ can we expect the stationary EDF to be close to $M(u)$. But in the physically interesting regimes it is ϵ that is small compared to E. On the other hand, if $\epsilon\simeq 0$ the collisions almost do not change the electron energy, so it is difficult for the electrons to get rid of the energy they acquire from the field. The limit $\epsilon\to 0$ is therefore singular. In particular, there is no well-defined reference stationary state for $\epsilon=0$ about which to expand the solution of Eq. (2).

B. Legendre expansion

The usual method [8] of solving Eq. (2) is to expand $f(\mathbf{u})$ in terms of the Legendre polynomials P_l,

$$f(\mathbf{u})=\sum_{l=0}^{\infty} f_l(u)P_l(\cos\ \theta), \quad (5)$$

$$f_l(u)=\frac{2l+1}{4\pi}\int f(\mathbf{u})P_l(\cos\ \theta)d\Omega_u,$$

where θ is the angle between $\mathbf{u}$ and the field $\mathbf{F}$: $\cos\theta=u_z/u$. Substituting Eq. (5) into Eq. (2) we obtain an infinite set of coupled ordinary differential equations for $l\geqslant 0$, $u\geqslant 0$. These have the form

$$-\frac{E}{3}\left(\frac{df_1}{du}+\frac{2}{u}f_1\right)=\epsilon\frac{1}{u^2}\frac{d}{du}\left[u^4\left(f_0+\frac{T}{u}\frac{df_0}{du}\right)\right], \quad l=0 \quad (6)$$

and

$$E\left[\frac{l}{2l-1}\left(\frac{df_{l-1}}{du}-\frac{l-1}{u}f_{l-1}\right)+\frac{l+1}{2l+3}\left(\frac{df_{l+1}}{du}+\frac{l+2}{u}f_{l+1}\right)\right]=uf_l, \quad l=1,2,\ldots\ . \quad (7)$$

Equation (6) can be integrated to give

$$f_1=-\frac{3\epsilon}{E}u^2\left(f_0+\frac{T}{u}\frac{df_0}{du}\right), \quad (8)$$

where the arbitrary constant of integration was taken to be 0, using reasonable assumptions on the behavior of f as $u\to 0$ and $u\to\infty$.

In the conventional [8–10] approximation scheme only two terms of expansion (5) are kept. This is equivalent to assuming $f_l(v)\equiv 0$ for $l\geqslant 2$. One then adds to Eq. (8) one more differential equation, obtained from Eq. (7), for $l=1$,

$$E\frac{df_0}{du}=uf_1. \quad (9a)$$

Substituting Eq. (8) into Eq. (9) then yields an equation for f_0,

$$\left(1+\frac{3\epsilon T}{E^2}u^2\right)\frac{df_0}{du}+\frac{3\epsilon}{E^2}u^3 f_0=0, \quad (9b)$$

whose solution is

$$f_0=C\exp\left(-\int_0^u\frac{x^3dx}{Tx^2+E^2/3\epsilon}\right). \quad (9c)$$

This f_0 becomes the Maxwellian $M(u)$ [Eq. (4)] when $E=0$ and the Druyvesteyn [11] distribution f^D when $T=0$:

$$f_0=f^D=C\exp\left(-\frac{3\epsilon u^4}{4E^2}\right),$$
$$C=\sqrt{2}\left(\frac{3\epsilon}{E^2}\right)^{3/4}\Big/\Gamma\left(\frac{3}{4}\right), \quad (10a)$$

where Γ is the Gamma function. Using Eqs. (9) and (10a) one can find f_1,

$$f_1=-C\frac{3\epsilon u^2}{E}\exp\left(-\frac{3\epsilon u^4}{4E^2}\right). \quad (10b)$$

For $T>0$, f_0 in Eq. (9c) will always have a Maxwellian form for $u\geqslant(E^2/T\epsilon)^{1/2}$.

The first two harmonics are sufficient to find the mean energy per particle W and mean speed (drift) w of the electrons, which are physically the most important properties of the stationary state,

$$W=\frac{m}{8\pi}\int v^2 f(v)d^3\mathbf{v}=\frac{m}{2\gamma^2}\int_0^\infty u^4 f_0 du,$$
$$w=\frac{-1}{4\pi\gamma^2}\int u_z f\, d^3u=\frac{-1}{3\gamma}\int_0^\infty u^3 f_1 du. \tag{11}$$

We shall now study the properties of these moments without the approximations made for explicitly solving Eq. (2).

III. MOMENTS OF THE DISTRIBUTION FUNCTION

We assume that moments

$$\mathcal{M}_k^{(l)}=\int_0^\infty u^k f_l(u)du \tag{12}$$

exist at least for $0\leqslant k\leqslant 9$. Multiplying Eq. (7) by a positive power k of u and integrating over u yields the equation

$$E\left[-l\,\frac{l+k-1}{2l-1}\,\mathcal{M}_{k-1}^{(l-1)}+\frac{(l+1)(l+2-k)}{2l+3}\,\mathcal{M}_{k-1}^{(l+1)}\right]=\mathcal{M}_{k+1}^{(l)}. \tag{13}$$

In terms of these moments w and W can be written, using Eqs. (11) and (8), as

$$w=\frac{\epsilon}{E\gamma}\,[\mathcal{M}_5^{(0)}-4T\mathcal{M}_3^{(0)}],\quad W=\frac{m}{2\gamma^2}\,\mathcal{M}_4^{(0)}. \tag{14}$$

We will now construct estimates of w and W by using Eqs. (8) and (13) to get relations between the $M_k^{(0)}$. (i) Taking $l=1$ and $k=3$ in Eq. (13) and substituting Eq. (8) for the calculation of $\mathcal{M}_4^{(1)}$ gives

$$\mathcal{M}_2^{(0)}=1=\frac{\epsilon}{E^2}\,(\mathcal{M}_6^{(0)}-5T\mathcal{M}_4^{(0)}). \tag{15}$$

(ii) For $l=1$, $k=6$, Eqs. (13) and (8) yield

$$\mathcal{M}_5^{(0)}+\frac{1}{5}\,\mathcal{M}_5^{(2)}=\frac{\epsilon}{2E^2}\,(\mathcal{M}_9^{(0)}-8T\mathcal{M}_7^{(0)}). \tag{16}$$

(iii) The set $l=2$, $k=4$ allows us to find $\mathcal{M}_5^{(2)}$,

$$\mathcal{M}_5^{(2)}=-\frac{10}{3}\,E\mathcal{M}_3^{(1)}=10\epsilon(\mathcal{M}_5^{(0)}-4T\mathcal{M}_3^{(0)})$$

and eliminate it from Eq. (16) to obtain

$$(1+2\epsilon)\mathcal{M}_5^{(0)}-8T\epsilon\mathcal{M}_3^{(0)}=\frac{\epsilon}{2E^2}\,(\mathcal{M}_9^{(0)}-8T\mathcal{M}_7^{(0)}). \tag{17}$$

Further calculation using different l and k will give additional equations for the $\mathcal{M}_j^{(0)}$, which might improve the estimates, but we shall use here only Eqs. (15) and (17).

Exploiting now general bounds on moments of the nonnegative density $f_0(u)$ derived in the Appendix, we obtain two-sided bounds for $\mathcal{M}_3^{(0)},\mathcal{M}_4^{(0)},\mathcal{M}_5^{(0)}$, which determine, by Eq. (14), the electron drift w and mean energy W.

Inequalities

The upper bounds on $\mathcal{M}_j$, $j=3,4,5$ (we have dropped the superscript zero), can be calculated from Eq. (15) using Eq. (A5):

$$\mathcal{M}_4\leqslant\mathcal{M}_6^{1/2}\Rightarrow 1\geqslant\frac{\epsilon}{E^2}\,(\mathcal{M}_4^2-5T\mathcal{M}_4)$$
$$\Rightarrow\mathcal{M}_4^2-5T\mathcal{M}_4-\frac{E^2}{\epsilon}\leqslant 0.$$

By solving the last inequality one gets

$$\mathcal{M}_4\leqslant a,\quad a=\frac{5T}{2}+\sqrt{\frac{E^2}{\epsilon}+\left(\frac{5T}{2}\right)^2}. \tag{18}$$

The same technique using bounds

$$\mathcal{M}_3\leqslant(\mathcal{M}_6)^{1/4},\quad\mathcal{M}_5\leqslant(\mathcal{M}_6)^{3/4}$$

gives

$$\mathcal{M}_3\leqslant a^{1/2},\quad\mathcal{M}_5\leqslant a^{3/2},\quad\mathcal{M}_6\leqslant a^2,\quad\frac{\mathcal{M}_6}{\mathcal{M}_4}\geqslant a. \tag{19}$$

The derivation of lower bounds via Eqs. (15) and (17) is more intricate. Keeping in mind that ϵ is small, we use Eq. (17) in the form of an inequality

$$\frac{2E^2}{\epsilon}\,(1+2\epsilon)>\frac{\mathcal{M}_9}{\mathcal{M}_5}-8T\,\frac{\mathcal{M}_7}{\mathcal{M}_5}\geqslant\sqrt{\frac{\mathcal{M}_9}{\mathcal{M}_5}}\left(\sqrt{\frac{\mathcal{M}_9}{\mathcal{M}_5}}-8T\right),$$

where we have used $\mathcal{M}_7\leqslant\sqrt{\mathcal{M}_5\mathcal{M}_9}$ in virtue of Eq. (A5). Using now Eq. (A6) with $j=5$, $n=1$, and $s=4$ we obtain

$$\frac{\mathcal{M}_9}{\mathcal{M}_5}\geqslant\left(\frac{\mathcal{M}_6}{\mathcal{M}_5}\right)^4$$

and a quadratic inequality for $\mathcal{M}_6/\mathcal{M}_5$ whose solution is

$$\frac{\mathcal{M}_6}{\mathcal{M}_5}\leqslant b^{1/2},\quad b=4T+\sqrt{(4T)^2+\frac{2E^2(1+2\epsilon)}{\epsilon}}. \tag{20}$$

We repeat now in Eq. (20) the use of Eq. (A6) with $i=6$, $k=1$, $s=2$ and $i=6$, $k=2$, $s=\frac{3}{2}$ with the results

$$\frac{\mathcal{M}_6}{\mathcal{M}_4}\leqslant b,\quad\frac{\mathcal{M}_6}{\mathcal{M}_3}\leqslant b^{3/2}. \tag{21}$$

One can solve Eq. (15) for $\mathcal{M}_6$ in terms of $\mathcal{M}_4$ and using Eq. (21) obtain the inequality

$$\mathcal{M}_4=\frac{\mathcal{M}_4}{\mathcal{M}_6}\,\mathcal{M}_6=\frac{\mathcal{M}_4}{\mathcal{M}_6}\left(\frac{E^2}{\epsilon}+5T\mathcal{M}_4\right)\geqslant b^{-1}\left(\frac{E^2}{\epsilon}+5T\mathcal{M}_4\right).$$

Its solution is

$$\mathcal{M}_4 \geqslant \frac{E^2}{\epsilon(b-5T)}. \tag{22}$$

Similarly expressing $\mathcal{M}_5$ and $\mathcal{M}_3$ through $\mathcal{M}_5/\mathcal{M}_6$ and $\mathcal{M}_3/\mathcal{M}_6$, respectively, and using Eqs. (15) and (20)–(22) we find the lower bounds. Together with Eq. (19) they allow us to write down two-sided bounds for $\mathcal{M}_j$ ($j=3,4,5$) in the form

$$a^{j/2-1} \geqslant \mathcal{M}_j \geqslant b^{j/2-2} \frac{E^2}{\epsilon(b-5T)}. \tag{23}$$

These are sufficient, by Eq. (14), for the estimation of w and W. One can write immediately

$$\frac{ma}{2\gamma^2} \geqslant W \geqslant \frac{mE^2}{2\gamma^2 \epsilon(b-5T)}. \tag{24a}$$

Using the definition (14) and the inequality (A5) we obtain

$$\frac{\epsilon}{E\gamma} \mathcal{M}_5 \geqslant w \geqslant \frac{\epsilon}{E\gamma} \mathcal{M}_5^{1/3}(\mathcal{M}_5^{2/3}-4T), \tag{24b}$$

which can be combined with Eq. (23) for $j=5$ to get explicit bounds on w.

The lower bounds in Eq. (23) are useless when $E \to 0$ and the solution of Eq. (2) approaches the Maxwellian. Generally, the inequalities (23) become more useful the larger E is.

IV. COMPARISON WITH THE DRUYVESTEYN APPROXIMATION

When the background temperature T is small compared with $E\epsilon^{-1/2}$ it can be neglected in Eqs. (18) and (20) and the bounds (24) look simpler:

$$\begin{gathered} \frac{\epsilon^{1/4}\sqrt{E}}{\gamma} \geqslant w \geqslant \frac{\epsilon^{1/4}\sqrt{E}}{\gamma[2(1+2\epsilon)]^{1/4}}, \\ \frac{mE}{2\gamma^2\sqrt{\epsilon}} \geqslant W \geqslant \frac{mE}{2\gamma^2\sqrt{2\epsilon(1+2\epsilon)}}. \end{gathered} \tag{25}$$

These bounds specify the electron drift and mean energy as functions of the electric field and gas parameters within errors of about $\pm 20\%$ for the mean energy and $\pm 8\%$ for the drift uniformly in E and ϵ. For comparison w and W obtained from the Druyvesteyn distribution (10a) are

$$w \approx 0.897 \frac{\epsilon^{1/4}E^{1/2}}{\gamma}, \quad W \approx 0.854 \frac{mE}{2\gamma^2\sqrt{\epsilon}}, \tag{26}$$

in good agreement with Eq. (25) when $\epsilon \ll 1$.

Experimentalists also measure sometimes the transversal D_t and longitudinal D_L diffusion constants for the electron swarm. While D_L cannot generally be expressed [2,9] in terms of the velocity moments,

$$D_t = D = \frac{\overline{\lambda}}{3\gamma} \mathcal{M}_3$$

is just the isotropic diffusion constant, where $\overline{\lambda}$ is the mean free path of electrons ($\overline{\lambda}=\lambda$ here). When T can be neglected we obtain

$$\frac{\lambda}{3\gamma}\left[\frac{E^2}{\epsilon}\right]^{1/4} \geqslant D \geqslant [2(1+2\epsilon)]^{-3/4} \frac{\lambda}{3\gamma}\left[\frac{E^2}{\epsilon}\right]^{1/4}. \tag{27}$$

For comparison

$$D \approx 0.759 \frac{\lambda}{3\gamma}\left(\frac{E^2}{\epsilon}\right)^{1/4}$$

in the Druyvesteyn approximation.

V. MOBILITY IN BINARY MIXTURES

The increase of electron mobility w/F in a plasma upon the addition of a small amount of a new gas has been observed in [3]. It was calculated in [4] within the two-term approximation (8) and (9) for binary mixtures of a heavy noble Ramsauer gas and helium addition. We shall show here rigorously that this effect exists even with constant collision cross sections. Using Eq. (11) gives

$$w = -\frac{1}{3\gamma} \mathcal{M}_3^{(1)} \tag{28}$$

and for $l=1$ Eq. (13) reads

$$\mathcal{M}_{k+1}^{(1)} = E\left(-k\mathcal{M}_{k-1}^{(0)} + 2\frac{3-k}{5}\mathcal{M}_{k-1}^{(2)}\right). \tag{29}$$

When $E \to 0$ we may neglect the second term in Eq. (29) and obtain

$$w \approx \frac{2E}{3\gamma} \mathcal{M}_1^{(0)} \approx \frac{4E}{3\gamma\sqrt{2\pi T}} = \frac{2}{3}\sqrt{\frac{2}{\pi}} \frac{eF\lambda}{\sqrt{mkT_n}}, \tag{30}$$

using Eq. (4) and the initial notation. The resistivity F/enw is here proportional to $\Sigma N_i \sigma_i$, which is just Mattiessen's rule.

Let us consider now the case of a strong field $kT_0 \ll eF\lambda/\sqrt{\epsilon}$ for a binary mixture $i=1,2$ and use the two-term ansatz (8) and (9). We then have the Druyvesteyn distribution (10) with the moments (26). Using Eq. (14) and the notation

$$\alpha = \frac{N_2}{N_1+N_2}, \quad \mu = \frac{M_1}{M_2}, \quad \theta = \frac{\sigma_2}{\sigma_1}.$$

we can write explicit expressions for the drift and mean electron energy

$$w = 0.897\sqrt{\frac{eF}{(N_1+N_2)\sigma_1\sqrt{mM_1}}} \frac{(1-\alpha+\alpha\theta\mu)^{1/4}}{(1-\alpha+\alpha\theta)^{3/4}}, \tag{31}$$

$$\begin{aligned} W = 0.427 \frac{eF}{(N_1+N_2)} \sqrt{\frac{M_1}{m}} [&(1-\alpha+\alpha\theta) \\ &\times(1-\alpha+\alpha\theta\mu)]^{-1/4}. \end{aligned} \tag{32}$$

Both the current and energy of electrons increase, but the mobility w/F decreases, as the field F increases.

Let us now keep the total gas density N_1+N_2 constant and vary the relative concentration of components by changing α. A simple analysis of Eq. (31) shows that w can be nonmonotonic when both θ and μ are larger than 1. For example, if $\theta=5$, $\mu=20$, then considering w as a function of α, $w=w(\alpha)$, we have

$$\frac{w(\alpha_m)}{w(0)}\approx 1.41, \quad \frac{w(1)}{w(0)}\approx 0.95.$$

Here $w(\alpha_m)$ is the maximum value of w obtained for $\alpha_m \approx 0.11$. The drift speed is almost the same in the pure species 1 and 2, but it is noticeably larger in a mixture. The mean energy of electrons changes more. When the lighter component substitutes for the heavier one it goes down:

$$\frac{W(\alpha_m)}{W(0)}\approx 0.46, \quad \frac{W(1)}{W(0)}\approx 0.21.$$

There is even a more striking situation, when one just adds the lighter gas keeping the density N_1 of the heavier component constant. In this case

$$w(\delta)\sim\frac{(1+\delta\theta\mu)^{1/4}}{(1+\delta\theta)^{3/4}},$$

$$W(\delta)\sim(1+\delta)^{-1/2}(1+\delta\theta)^{-1/4}(1+\delta\theta\mu)^{-1/4}, \qquad (33)$$

where $\delta=N_2/N_1$. Increasing δ, we increase the density of scatterers, but for $\delta=\delta_m=8.5\%$

$$\frac{w(\delta_m)}{w(0)}\approx 1.4,$$

while the electron energy decreases: $W(\delta_m)\approx 0.5W(0)$.

We obtained these results approximately, by truncating the series (5). However, comparing Eq. (26) with the bounds (24), we see that the drift velocity and mean energy for the Druyvesteyn approximation cannot differ from the exact solution by more than about $+12\%$, -6%, and $\pm 17\%$, respectively. Hence the nonmonotonic dependence of the electron mobility on the density of the light species holds for the exact solution of the kinetic equation (2). When we had $w_{\max}\approx 1.40w(0)$ (within the approximation) a possible exaggeration of $w_{\max}$ by 12% and underestimation of $w(0)$ of at most 6% could reduce their ratio from 1.40 to 1.16, but the effect is clearly there without approximations.

The explanation of such unusual behavior of the electron drift in the nonlinear regime is quite simple. When $M_2 < M_1$ the addition of species 2 makes the energy transfer from the electrons to atoms easier in the elastic collisions. Consequently, the mean electron energy W will drop leading to a net increase of the mean free time $\tau(v)\sim\lambda/v$. The competition of λ and v is shown by formulas (31) and (33), where α,δ represent the concentration of the lighter species and μ is proportional to its relative effectiveness in the energy transfer. Adding about 10% of a component with atoms of mass $m_2\sim 0.05m_1$, the mean electron energy decreases by about $\frac{1}{2}$, implying the increase of w by about 40%. This rise of the electron mobility can be stronger [4] in the case when the collision cross section of the main (heavy) component is energy dependent and decreases with the electron energy.

VI. SIMPLE VELOCITY-DEPENDENT COLLISION CROSS SECTIONS

We consider here a one-species plasma with the atoms of mass M and generalize the bounds (24) for the e-n collision cross section of the form

$$\sigma(v)=\sigma_0\left(\frac{v}{v_0}\right)^p, \qquad (34)$$

where the exponent p can be positive or negative in a certain range. Setting

$$v_0^2=\frac{eF}{mN\sigma_0}, \quad t=\epsilon^{1/2}\frac{kT_n}{mv_0^2}, \quad \epsilon=\frac{m}{M},$$

we can rewrite Eq. (1) as

$$-\epsilon^{(p+2)/4}\frac{\partial f}{\partial y_z}=\epsilon\frac{1}{y^2}\frac{d}{dy}\left[y^{p+4}\left(f_0+\frac{t}{y}\frac{df_0}{dy}\right)\right] + y^{p+1}(f_0-f), \qquad (35)$$

where $v=\epsilon^{-1/4}v_0 y$ and we have in mind situations with ''strong'' electric field $t\ll 1$. Using the Legendre series expansion (5) for $f(\mathbf{y})$, we again obtain the infinite set of coupled equations for harmonics $f_l(y)$,

$$\epsilon^{-(p+2)/4}y^{1+p}f_l=\frac{l}{2l+1}\left(\frac{df_{l-1}}{dy}-\frac{l-1}{y}f_{l-1}\right) + \frac{l+1}{2l+3}\left(\frac{df_{l+1}}{dy}+\frac{l+2}{y}f_{l+1}\right) \qquad (36)$$

for $l=1,2,3,\ldots$ and one more equation

$$f_1=-3\epsilon^{(2-p)/4}y^{p+2}\left(f_0+\frac{t}{y}\frac{df_0}{dy}\right), \qquad (37)$$

corresponding to Eq. (8).

Methods similar to those in Sec. II allow us to derive the pair of equations for moments, which generalize Eqs. (16) and (17):

$$\mathcal{M}(2p+6)=\epsilon^{p/2}\mathcal{M}(2), \quad \mathcal{M}(3p+9)=\epsilon^{p/2}c\mathcal{M}(p+5), \qquad (38)$$

where

$$c=\tfrac{1}{3}[p+6+4\epsilon(p+3)], \quad \mathcal{M}(k)=\int_0^\infty f_0(y)y^k dy,$$

and the background temperature parameter t is neglected for simplicity. In terms of these moments, which clearly satisfy Eq. (A2), we have for the electron drift and mean energy

$$w=\epsilon^{(1-p)/4}v_0\mathcal{M}(p+5), \quad W=\epsilon^{-1/2}\frac{mv_0^2}{2}\mathcal{M}(4). \qquad (39)$$

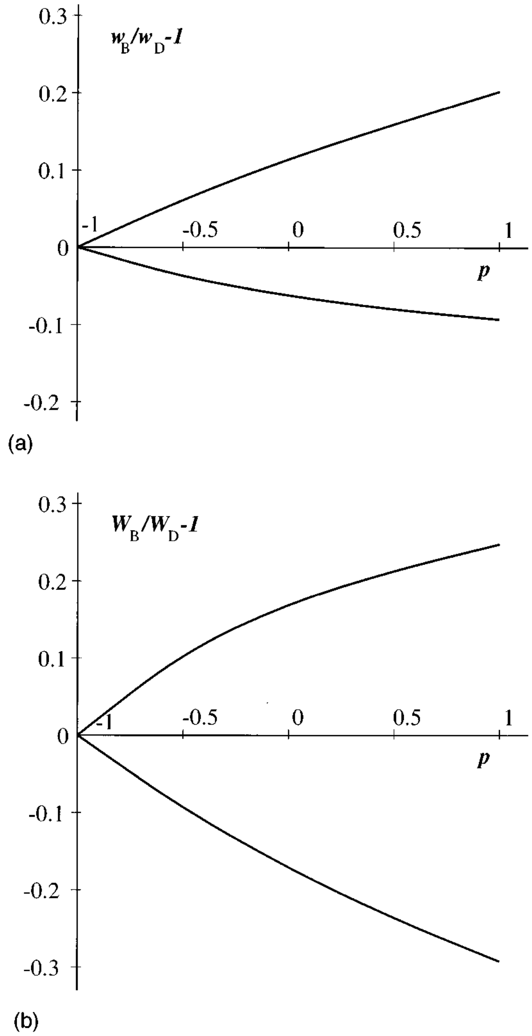

FIG. 1. Bounds of the (a) electron drift and (b) mean energy as functions of the exponent p in Eq. (34).

A calculation similar to that described in Sec. II and the Appendix shows that Eqs. (38) and (39) yield the following upper (U) and lower (L) bounds for w and W:

$$w_L \leqslant w \leqslant w_U, \quad W_L \leqslant W \leqslant W_U,$$

$$w_L = v_0 \left(\frac{\epsilon}{c} \right)^{(p+1)/(2p+4)}, \quad w_U = v_0 \epsilon^{(p+1)/(2p+4)}, \tag{40}$$

$$W_L = \frac{m v_0^2}{2} \epsilon^{-1/(p+2)} c^{-(p+1)/(p+2)}, \quad W_U = \frac{m v_0^2}{2} \epsilon^{-1/(p+2)},$$

which give Eq. (24) for the velocity independent cross section $p=0$ when $T \ll \epsilon^{-1/2} E$.

We can find the approximate solution of Eq. (35)

$$f_0^D(y) = C \exp\left[-3 \int_0^y \frac{x^{2p+3} dx}{\epsilon^{p/2} + 3 t x^{2+2p}} \right], \tag{41}$$

using the two-term ansatz that leads to the Druyvesteyn function (9c) for $p=0$. Computing the moments in Eq. (39) with the help of Eq. (41) yields the explicit formulas

$$w_D = \epsilon^{(p+1)/(2p+4)} v_0 \left[\frac{2p+4}{3} \right]^{(p+3)/(2p+4)} \times \Gamma\left(\frac{p+6}{2p+4} \right) \Big/ \Gamma\left(\frac{3}{2p+4} \right),$$

$$W_D = \epsilon^{-1/(p+2)} \frac{m v_0^2}{2} \left[\frac{2p+4}{3} \right]^{1/(p+2)} \times \Gamma\left(\frac{5}{2p+4} \right) \Big/ \Gamma\left(\frac{3}{2p+4} \right). \tag{42}$$

The bounds in Eq. (40) for the drift and energy as functions of the parameter p are shown in Fig. 1 in the form $w_B/w_D - 1$, $W_B/W_D - 1$, respectively, with the Druyvesteyn result (42) for comparison (we use the subscript "B" for both "L" and "U"). The accuracy of two-term approximation for our models is quite good.

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APPENDIX

The moments $\mathcal{M}_k$ involved in Eqs. (15) and (17)–(24) are the integrals of the non-negative function $f_0(u)$:

$$f_0(u) = \tfrac{1}{2} \int_0^\pi f(\mathbf{u}) \sin\, \theta \, d\theta.$$

We can easily show that $\ln \mathcal{M}(k)$ is a concave function if one treats k as a continuous variable:

$$\frac{d^2}{dk^2} \ln \mathcal{M} \geqslant 0. \tag{A1}$$

Equation (A1) is equivalent to the inequality

$$\mathcal{M} \frac{d^2 \mathcal{M}}{dk^2} \geqslant \left(\frac{d\mathcal{M}}{dk} \right)^2, \tag{A2}$$

which can be written using Eq. (12) as

$$\int_0^\infty x^k f_0(x) dx \int_0^\infty y^k \ln^2(y) f_0(y) dy - \left(\int_0^\infty x^k \ln x f_0(x) dx \right)^2$$
$$= \tfrac{1}{2} \int_0^\infty \int_0^\infty x^k y^k \ln^2\left(\frac{x}{y} \right) f_0(x) f_0(y) dx\, dy \geqslant 0.$$

The concavity implies obviously

$$\frac{\ln\mathcal{M}_k-\ln\mathcal{M}_i}{k-i}\leqslant\frac{\ln\mathcal{M}_n-\ln\mathcal{M}_m}{n-m}, \tag{A3}$$

$$k>i\geqslant 0,\quad n>m\geqslant i,\quad n\geqslant k.$$

Taking $k-i=n-m$, $n-k=j$ we obtain

$$\frac{\mathcal{M}_k}{\mathcal{M}_i}\leqslant\frac{\mathcal{M}_{k+j}}{\mathcal{M}_{i+j}},\quad k>i,\quad j>0. \tag{A4}$$

For the case $k=m$ Eq. (A3) yields the inequality

$$(\mathcal{M}_k)^{j-i}\leqslant(\mathcal{M}_i)^{j-k}(\mathcal{M}_j)^{k-i},\quad 0\leqslant i<k<j, \tag{A5}$$

which is equivalent to the useful set

$$\left(\frac{\mathcal{M}_{j+n}}{\mathcal{M}_j}\right)^s\leqslant\frac{\mathcal{M}_{j+sn}}{\mathcal{M}_j},\quad \left(\frac{\mathcal{M}_i}{\mathcal{M}_{i-k}}\right)^s\geqslant\frac{\mathcal{M}_i}{\mathcal{M}_{i-sk}}, \tag{A6}$$

where $i,j,n,k\geqslant 0$, $s\geqslant 1$, and $i\geqslant sk$.

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